All-Pairs Shortest Paths

(The Floyd-Warshall Algorithm)

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- Floyd-Warshall
  - Shortest path between **all** pair of vertices
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- We assume that 'G' doesn't contain any negative weight cycle.
- The All-Pairs Shortest Path problem asks to find the length of the shortest path between any pair of vertices in 'G'.

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- Linear array implementation of min-priority queue: $O(V^3 + VE)$
- Fibonacci Heap implementation of min-priority queue: $O(V^2 logV + VE)$

Solutions Using Previous Knowledge (Cont...)

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CAN WE DO BETTER ?

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Alert

Negative-weight edges may be present, but we assume that "no Negative weight cycles" ..

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$$\boldsymbol{w_{ij}} = \begin{cases} 0, & \text{if } i = j \\ w(i,j), & \text{if } i \neq j \text{ and} (i,j) \text{ in } E \\ \infty, & \text{if } i \neq j \text{ and} (i,j) \text{ not in } E \end{cases}$$

Format of the Output

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- If the graph has **V** vertices, we return a distance matrix **D**, where each element **(**d_{ij}**)** is the shortest length of the path from i to j.

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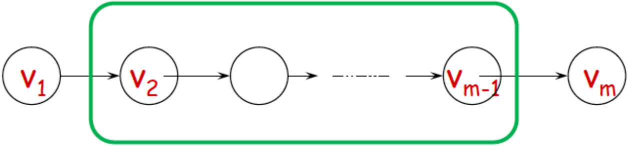

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- By assumption, cycles in the graph have a positive weight. So removing the cycle would result in a shorter path.

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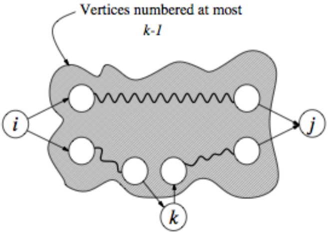

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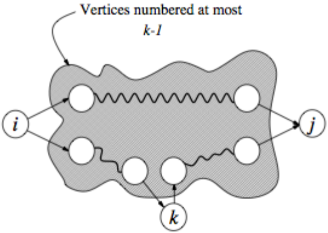

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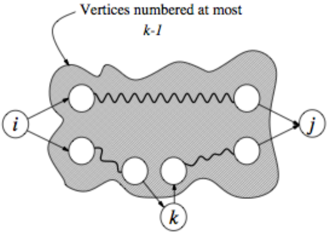

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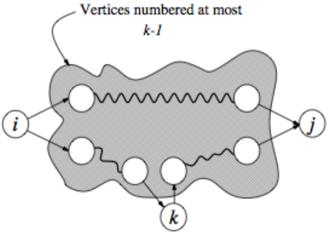

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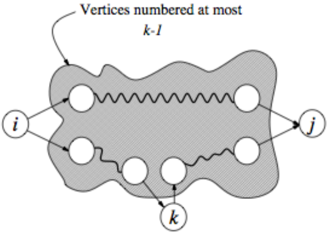

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- Therefore, we can conclude that :

$$\boldsymbol{d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}}$$

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- Mathematically,

$$\boldsymbol{d_{ij}^{(k)}} = \begin{cases} w_{ij}, & \text{if } k = 0 \\ \text{min}\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}, & \text{if } k > 0 \end{cases}$$

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FLOYD-WARSHALL(W)

```
n ← rows[W]
D^(0) ← W
for k ← 1 to n
        do for i ← 1 to n
                do for j ← 1 to n
```

The Floyd-Warshall Algorithm

FLOYD-WARSHALL(W)

$n \leftarrow rows[W]$
$D^{(0)} \leftarrow W$
for $k \leftarrow 1$ **to** n
  do for $i \leftarrow 1$ **to** n
    do for $j \leftarrow 1$ **to** n
      do $d_{ij}^{(k)} \leftarrow min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$
return $D^{(n)}$

Time and Space Requirements

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